

ON NEW INEQUALITIES OF HERMITE-HADAMARD-FEJER TYPE FOR CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

ERHAN SET[♦], İMDAT İŞCAN[▼], M. ZEKI SARIKAYA[▲], AND M. EMİN ÖZDEMİR[■]

ABSTRACT. In this paper, we establish some weighted fractional inequalities for differentiable mappings whose derivatives in absolute value are convex. These results are connected with the celebrated Hermite-Hadamard-Fejér type integral inequality. The results presented here would provide extensions of those given in earlier works.

1. INTRODUCTION

Throughout this paper, let I be an interval on \mathbb{R} and let $\|g\|_{[a,b],\infty} = \sup_{t \in [a,b]} |g(t)|$, for the continuous function $g : [a, b] \rightarrow \mathbb{R}$.

Let $f : I \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality holds:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions [7].

In order to prove some inequalities related to Hermite Hadamard inequality, Kırmacı used the following lemma:

Lemma 1. ([12]) *Let $f : I^\circ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then we have*

$$(1.2) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \\ &= (b-a) \int_0^{\frac{1}{2}} t f'(ta + (1-t)b) dt + \int_{\frac{1}{2}}^1 (t-1) f'(ta + (1-t)b) dt. \end{aligned}$$

Theorem 1. ([12]) *Let $f : I^\circ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$(1.3) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|).$$

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Theorem 2. ([12]) *Let $f : I^\circ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{p/p-1}$ is convex on $[a, b]$, then we have*

$$(1.4) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{\frac{1}{p}} \left[\left(|f'(a)|^{p/p-1} + 3|f'(b)|^{p/p-1} \right)^{(p-1)/p} + \left(3|f'(a)|^{p/p-1} + |f'(b)|^{p/p-1} \right)^{(p-1)/p} \right].$$

The most well known inequalities connected with the integral mean of a convex functions are Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities. In [6], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1).

Theorem 3. *Let $f : I \rightarrow \mathbb{R}$ be a convex on I and let $a, b \in I$ with $a < b$. Then the inequality*

$$(1.5) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx \leq \int_a^b f(x)g(x) dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x) dx$$

holds, where $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric to $\frac{a+b}{2}$.

In [13], Sarikaya established some inequalities of Hermite-Hadamard-Fejér type for differentiable convex functions using the following lemma:

Lemma 2. *Let $f : I^\circ \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$, and $g : [a, b] \rightarrow [0, \infty)$ be a differentiable mapping. If $f' \in L[a, b]$, then the following identity holds:*

$$(1.6) \quad \frac{1}{b-a} \int_a^b f(x)g(x) dx - \frac{1}{b-a} f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx = (b-a) \int_0^1 k(t) f'(ta+(1-t)b) dt$$

for each $t \in [0, 1]$, where

$$k(t) = \begin{cases} \int_0^1 w(as+(1-s)b) ds, & t \in [0, \frac{1}{2}) \\ -\int_0^1 w(as+(1-s)b) ds, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Meanwhile, in [16] Sarikaya and Erden gave the following interesting identity and by using this identity they established some interesting integral inequalities:

Lemma 3. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and let $w : [a, b] \rightarrow \mathbb{R}$. If $f', w \in L[a, b]$, then, for all $x \in [a, b]$, the following*

equality holds:

$$\begin{aligned}
 (1.7) \quad & \int_a^x \left(\int_a^t w(s) ds \right)^\alpha f'(t) dt - \int_x^b \left(\int_t^b w(s) ds \right)^\alpha f'(t) dt \\
 &= \left[\left(\int_a^x w(s) ds \right)^\alpha + \left(\int_x^b w(s) ds \right)^\alpha \right] f(x) \\
 &\quad - \alpha \int_a^x \left(\int_a^t w(s) ds \right)^{\alpha-1} w(t) f(t) dt - \alpha \int_x^b \left(\int_t^b w(s) ds \right)^{\alpha-1} w(t) f(t) dt.
 \end{aligned}$$

For several recent results concerning inequality (1.5), see [8], [13], [16], [17], [19] where further references are listed.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1. Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. Here is $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In [15], Sarikaya et. al. represented Hermite-Hadamard's inequalities in fractional integral forms as follows.

Theorem 4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold

$$(1.8) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

In [8], İşcan gave the following Hermite-Hadamard-Fejer integral inequalities via fractional integrals:

Theorem 5. Let $f : [a, b] \rightarrow \mathbb{R}$ be convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals hold

$$\begin{aligned}
 (1.9) \quad f\left(\frac{a+b}{2}\right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] &\leq [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \\
 &\leq \frac{f(a) + f(b)}{2} [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)]
 \end{aligned}$$

with $\alpha > 0$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals that are not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see ([1]-[5]), ([8]-[11]), ([14]-[18]).

The aim of this paper is to present some new Hermite-Hadamard-Fejér type results for differentiable mappings whose derivatives in absolute value are convex. The results presented here would provide extensions of those given in earlier works.

2. MAIN RESULTS

We establish some new results connected with the left-hand side of (1.5) used the following Lemma. Now, we give the following new Lemma for our results.

Lemma 4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and let $g : [a, b] \rightarrow \mathbb{R}$. If $f', g \in L[a, b]$, then the following identity for fractional integrals holds:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} g(b) \right] \\ & - \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} (fg)(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} (fg)(b) \right] \\ (2.1) \quad & = \frac{1}{\Gamma(\alpha)} \int_a^b k(t) f'(t) dt, \end{aligned}$$

where

$$k(t) = \begin{cases} \int_a^t (s-a)^{\alpha-1} g(s) ds & t \in [a, \frac{a+b}{2}] \\ \int_b^t (b-s)^{\alpha-1} g(s) ds & t \in [\frac{a+b}{2}, b] \end{cases}.$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_a^b k(t) f'(t) dt \\ &= \int_a^{\frac{a+b}{2}} \left(\int_a^t (s-a)^{\alpha-1} g(s) ds \right) f'(t) dt + \int_{\frac{a+b}{2}}^b \left(\int_b^t (b-s)^{\alpha-1} g(s) ds \right) f'(t) dt \\ &= I_1 + I_2. \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} I_1 &= \left(\int_a^t (s-a)^{\alpha-1} g(s) ds \right) f(t) \Big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} (t-a)^{\alpha-1} g(t) f(t) dt \\ &= \left(\int_a^{\frac{a+b}{2}} (s-a)^{\alpha-1} g(s) ds \right) f\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} (t-a)^{\alpha-1} (fg)(t) dt \\ &= \Gamma(\alpha) \left[f\left(\frac{a+b}{2}\right) J_{\left(\frac{a+b}{2}\right)-}^{\alpha} g(a) - J_{\left(\frac{a+b}{2}\right)-}^{\alpha} (fg)(a) \right], \end{aligned}$$

and similarly

$$\begin{aligned}
 I_2 &= \left(\int_b^t (b-s)^{\alpha-1} g(s) ds \right) f(t) \Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b (b-t)^{\alpha-1} g(t) f(t) dt \\
 &= \left(\int_{\frac{a+b}{2}}^b (b-s)^{\alpha-1} g(s) ds \right) f\left(\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^b (b-t)^{\alpha-1} (fg)(t) dt \\
 &= \Gamma(\alpha) \left[f\left(\frac{a+b}{2}\right) J_{\left(\frac{a+b}{2}\right)^+}^\alpha g(b) - J_{\left(\frac{a+b}{2}\right)^+}^\alpha (fg)(b) \right].
 \end{aligned}$$

Thus, we can write

$$\begin{aligned}
 I &= I_1 + I_2 \\
 &= \Gamma(\alpha) \left\{ f\left(\frac{a+b}{2}\right) \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha g(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha g(b) \right] - \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha (fg)(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha (fg)(b) \right] \right\}.
 \end{aligned}$$

Multiplying the both sides by $(\Gamma(\alpha))^{-1}$, we obtain (2.1) which completes the proof. \square

Remark 1. If we choose $\alpha = 1$ in Lemma 4, then the inequality (2.1) reduces to (1.6).

Now, we are ready to state and prove our results.

Theorem 6. Let $f : I \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$ and $g : [a, b] \rightarrow \mathbb{R}$ is continuous. If $|f'|$ is convex on $[a, b]$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha g(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha g(b) \right] \right. \\
 &\quad \left. - \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha (fg)(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha (fg)(b) \right] \right| \\
 (2.2) \quad &\leq \frac{(b-a)^{\alpha+1} \|g\|_{[a,b],\infty}}{2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)} (|f'(a)| + |f'(b)|)
 \end{aligned}$$

with $\alpha > 0$.

Proof. Since $|f'|$ is convex on $[a, b]$, we know that for $t \in [a, b]$

$$|f'(t)| = \left| f' \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| \leq \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)|.$$

From Lemma 4 we have

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} g(b) \right] - \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} (fg)(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} (fg)(b) \right] \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left\{ \int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} g(s) ds \right| |f'(t)| dt + \int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} g(s) ds \right| |f'(t)| dt \right\} \\
& \leq \frac{\|g\|_{[a, \frac{a+b}{2}], \infty}}{(b-a)\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left(\int_a^t (s-a)^{\alpha-1} ds \right) ((b-t)|f'(a)| + (t-a)|f'(b)|) dt \\
& \quad + \frac{\|g\|_{[\frac{a+b}{2}, b], \infty}}{(b-a)\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \left(\int_t^b (b-s)^{\alpha-1} ds \right) ((b-t)|f'(a)| + (t-a)|f'(b)|) dt \\
& = \frac{\|g\|_{[a, \frac{a+b}{2}], \infty}}{(b-a)\Gamma(\alpha+1)} \int_a^{\frac{a+b}{2}} (t-a)^{\alpha} ((b-t)|f'(a)| + (t-a)|f'(b)|) dt \\
& \quad + \frac{\|g\|_{[\frac{a+b}{2}, b], \infty}}{(b-a)\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^b (b-t)^{\alpha} ((b-t)|f'(a)| + (t-a)|f'(b)|) dt \\
& = \frac{(b-a)^{\alpha+1}}{2^{\alpha+2}(\alpha+2)(\alpha+1)\Gamma(\alpha+1)} \left\{ \|g\|_{[a, \frac{a+b}{2}], \infty} ((\alpha+3)|f'(a)| + (\alpha+1)|f'(b)|) \right. \\
& \quad \left. + \|g\|_{[\frac{a+b}{2}, b], \infty} ((\alpha+1)|f'(a)| + (\alpha+3)|f'(b)|) \right\} \\
& \leq \frac{(b-a)^{\alpha+1} \|g\|_{[a, b], \infty}}{2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)} (|f'(a)| + |f'(b)|)
\end{aligned}$$

where

$$\begin{aligned}
\int_a^{\frac{a+b}{2}} (t-a)^{\alpha+1} dt &= \int_{\frac{a+b}{2}}^b (b-t)^{\alpha+1} dt = \frac{(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}, \\
\int_a^{\frac{a+b}{2}} (t-a)^{\alpha} (b-t) dt &= \int_{\frac{a+b}{2}}^b (b-t)^{\alpha} (t-a) dt \\
&= \frac{(\alpha+3)(b-a)^{\alpha+2}}{2^{\alpha+2}(\alpha+1)(\alpha+2)}
\end{aligned}$$

This completes the proof. \square

Remark 2. If we choose $g(x) = 1$ and $\alpha = 1$ in Theorem 6, then the inequality (2.2) reduces to (1.3).

Theorem 7. Let $f : I \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$ and let $g : [a, b] \rightarrow \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality for fractional integrals holds:

$$(2.3) \quad \left| f\left(\frac{a+b}{2}\right) \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a) \right] - \left[J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a) \right] \right|$$

$$\begin{aligned}
 &\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1+\frac{1}{q}} (\alpha+1) (\alpha+2)^{1/q} \Gamma(\alpha+1)} \\
 &\quad \times \left\{ \|g\|_{[a, \frac{a+b}{2}], \infty} ((\alpha+3) |f'(a)|^q + (\alpha+1) |f'(b)|^q)^{1/q} \right. \\
 &\quad \left. + \|g\|_{[\frac{a+b}{2}, b], \infty} ((\alpha+1) |f'(a)|^q + (\alpha+3) |f'(b)|^q)^{1/q} \right\} \\
 &\leq \frac{(b-a)^{\alpha+1} \|g\|_{[a, b], \infty}}{2^{\alpha+1+\frac{1}{q}} (\alpha+1) (\alpha+2)^{1/q} \Gamma(\alpha+1)} \\
 &\quad \times \left\{ (|f'(a)|^q + (\alpha+1) |f'(b)|^q)^{1/q} \right. \\
 &\quad \left. + ((\alpha+1) |f'(a)|^q + |f'(b)|^q)^{1/q} \right\}
 \end{aligned}$$

with $\alpha > 0$.

Proof. Since $|f'|^q$ is convex on $[a, b]$, we know that for $t \in [a, b]$

$$|f'(t)|^q = \left| f' \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right|^q \leq \frac{b-t}{b-a} |f'(a)|^q + \frac{t-a}{b-a} |f'(b)|^q.$$

Using Lemma 4, Power mean inequality and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}
 &\left| f \left(\frac{a+b}{2} \right) \left[J_{\left(\frac{a+b}{2} \right)-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2} \right)+}^{\alpha} g(b) \right] - \left[J_{\left(\frac{a+b}{2} \right)-}^{\alpha} (fg)(a) + J_{\left(\frac{a+b}{2} \right)+}^{\alpha} (fg)(b) \right] \right| \\
 &\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} g(s) ds \right| dt \right)^{1-1/q} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} g(s) ds \right| |f'(t)|^q dt \right)^{1/q} \\
 &\quad + \frac{1}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} g(s) ds \right| dt \right)^{1-1/q} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} g(s) ds \right| |f'(t)|^q dt \right)^{1/q} \\
 &\leq \frac{\|g\|_{[a, \frac{a+b}{2}], \infty}}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} ds \right| dt \right)^{1-1/q} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} ds \right| |f'(t)|^q dt \right)^{1/q} \\
 &\quad + \frac{\|g\|_{[\frac{a+b}{2}, b], \infty}}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} ds \right| dt \right)^{1-1/q} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} ds \right| |f'(t)|^q dt \right)^{1/q}
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\alpha \Gamma(\alpha)} \left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)} \right)^{1-1/q} \\
&\quad \times \left\{ \frac{\|g\|_{[a, \frac{a+b}{2}], \infty}}{b-a} \left(\int_a^{\frac{a+b}{2}} (t-a)^\alpha (b-t) |f'(a)|^q + (t-a)^{\alpha+1} |f'(b)|^q dt \right)^{1/q} \right. \\
&\quad \left. + \frac{\|g\|_{[\frac{a+b}{2}, b], \infty}}{(b-a)^{1/q}} \left(\int_{\frac{a+b}{2}}^b (b-t)^{\alpha+1} |f'(a)|^q + (b-t)^\alpha (t-a) |f'(b)|^q dt \right)^{1/q} \right\} \\
&\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+\frac{1}{q}}(\alpha+1)(\alpha+2)^{1/q} \Gamma(\alpha+1)} \left\{ \|g\|_{[a, \frac{a+b}{2}], \infty} ((\alpha+3) |f'(a)|^q + (\alpha+1) |f'(b)|^q dt)^{1/q} \right. \\
&\quad \left. + \|g\|_{[\frac{a+b}{2}, b], \infty} ((\alpha+1) |f'(a)|^q + (\alpha+3) |f'(b)|^q dt)^{1/q} \right\} \\
&\leq \frac{(b-a)^{\alpha+1} \|g\|_{[a, b], \infty}}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+2)^{1/q} \Gamma(\alpha+1)} \left\{ ((\alpha+3) |f'(a)|^q + (\alpha+1) |f'(b)|^q dt)^{1/q} \right. \\
&\quad \left. + ((\alpha+1) |f'(a)|^q + (\alpha+3) |f'(b)|^q dt)^{1/q} \right\}
\end{aligned}$$

where it is easily seen that

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} ds \right| dt = \int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} ds \right| dt \\
&= \frac{(b-a)^{\alpha+1}}{2^{\alpha+1} \alpha (\alpha+1)}.
\end{aligned}$$

Hence, the proof is completed. \square

We can state another inequality for $q > 1$ as follows:

Theorem 8. *Let $f : I \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$ and let $g : [a, b] \rightarrow \mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) \left[J_{(\frac{a+b}{2})_-}^\alpha g(a) + J_{(\frac{a+b}{2})_+}^\alpha g(b) \right] \right. \\
&\quad \left. - \left[J_{(\frac{a+b}{2})_-}^\alpha (fg)(a) + J_{(\frac{a+b}{2})_+}^\alpha (fg)(b) \right] \right| \\
(2.4) \quad &\leq \frac{\|g\|_\infty (b-a)^{\alpha+1}}{2^{\alpha+1+\frac{2}{q}} (\alpha p + 1)^{1/p} \Gamma(\alpha+1)} \\
&\quad \times \left[(3 |f'(a)|^q + |f'(b)|^q)^{1/q} + (|f'(a)|^q + 3 |f'(b)|^q)^{1/q} \right]
\end{aligned}$$

where $1/p + 1/q = 1$.

Proof. Using Lemma 4, Hölder's inequality and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} g(b) \right] - \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} (fg)(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} (fg)(b) \right] \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} g(s) ds \right|^p dt \right)^{1/p} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q dt \right)^{1/q} \\
& \quad + \frac{1}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} g(s) ds \right|^p dt \right)^{1/p} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q dt \right)^{1/q} \\
& \leq \frac{(b-a)^{\frac{1}{q}} \|g\|_{[a, \frac{a+b}{2}], \infty}}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} ds \right|^p dt \right)^{1/p} \left[\frac{3|f'(a)|^q + |f'(b)|^q}{8} \right]^{1/q} \\
& \quad + \frac{(b-a)^{\frac{1}{q}} \|g\|_{[\frac{a+b}{2}, b], \infty}}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b \left| \int_b^t (b-s)^{\alpha-1} ds \right|^p dt \right)^{1/p} \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{8} \right]^{1/q} \\
& \leq \frac{\|g\|_{\infty} (b-a)^{\alpha+1}}{2^{\alpha+1+\frac{2}{q}} (\alpha p + 1)^{1/p} \Gamma(\alpha + 1)} \left[(3|f'(a)|^q + |f'(b)|^q)^{1/q} + (|f'(a)|^q + 3|f'(b)|^q)^{1/q} \right].
\end{aligned}$$

Here we use

$$\begin{aligned}
& \int_a^{\frac{a+b}{2}} \left| \int_a^t (s-a)^{\alpha-1} ds \right|^p dt = \frac{(b-a)^{\alpha p + 1}}{2^{\alpha p + 1} (\alpha p + 1) \alpha^p}, \\
& \int_a^{\frac{a+b}{2}} |f'(t)|^q dt \leq \frac{1}{b-a} \int_a^{\frac{a+b}{2}} [(b-t)|f'(a)|^q + (t-a)|f'(b)|^q] dt \\
& \quad = (b-a) \frac{3|f'(a)|^q + |f'(b)|^q}{8}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\frac{a+b}{2}}^b |f'(t)|^q dt \leq \frac{1}{b-a} \int_{\frac{a+b}{2}}^b [(b-t)|f'(a)|^q + (t-a)|f'(b)|^q] dt \\
& \quad = (b-a) \frac{|f'(a)|^q + 3|f'(b)|^q}{8}.
\end{aligned}$$

Hence the inequality (2.4) is proved. \square

Remark 3. If we choose $g(x) = 1$ and $\alpha = 1$ in Theorem 8, then the inequality (2.4) reduces to (1.4).

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[♦]DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, ORDU UNIVERSITY, 52200, ORDU, TURKEY

E-mail address: `erhanset@yahoo.com`

[▼]DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, GİRESUN UNIVERSITY, 28100, GİRESUN, TURKEY.

E-mail address: `imdat.iscan@giresun.edu.tr`, `imdati@yahoo.com`

[▲]DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, DÜZCE UNIVERSITY, 52200, DÜZCE, TURKEY

E-mail address: `sarikayamz@gmail.com`

[■]ATATÜRK UNIVERSITY, K.K. EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS, 25240, CAMPUS, ERZURUM, TURKEY

E-mail address: `emos@atauni.edu.tr`